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T.H. KOORNWINDER
THE ADDITION FORMULA FOR JACOBI
POLYNOMIALS AND SPHERICAL HARMONICS

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Abstract

Recently Gegenbauer's addition formula was generalized for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ by an algebraic approach (cf. the author's paper in Indag. Math. 34 (1972), pp. 188-191). Here a new algebraic proof using spherical harmonics will be presented. Let $q \leq p$ and let $e_1, e_2, \ldots, e_{q+p}$ be an orthonormal base of $R^{q+p}$ with unit sphere $\Omega$. The Jacobi polynomials $P_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(x)$ can be characterized as spherical harmonics of degree 2n on $\Omega$ which are invariant under the subgroup $SO(q) \times SO(p)$ of the rotation group $SO(q+p)$. Let the rotations $A_\tau$ be defined by $A_\tau\, e_k = \cos\tau\, e_k - \sin\tau\, e_{q+k}$ $(k=1,\ldots,q)$, $A_\tau\, e_{q+k} = \sin\tau\, e_k + \cos\tau\, e_{q+k}$ $(k=1,\ldots,q)$, $A_\tau\, e_k = e_k$ $(k=2q+1,\ldots,q+p)$. An explicit orthonormal base will be constructed for the set of those spherical harmonics of degree 2n which are invariant under all $T \in SO(q) \times SO(p)$ which commute with the rotations $A_\tau$. Let this base consist of the functions $S_k$ $(k=1,\ldots,N)$, then the kernel function $\Phi(\xi,\eta) = \sum_{k=1}^{N} S_k(\xi)\, S_k(\eta)$ $(\xi,\eta\in\Omega)$ satisfies $\Phi(\xi, A_\tau^{-1} e_1) = \Phi(A_\tau \xi, e_1)$. The addition formula for Jacobi polynomials $P_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(x)$ is finally obtained by writing this last identity in an explicit way.

*Remark*. This paper was presented by invitation at the Drexel University - SIAM Conference on "Lie algebras: applications and computational methods", held in Philadelphia, June 15 and 16, 1972. It will be published in the Proceedings of this Conference.

## 1. Introduction

Many classes of special functions can be considered as generalizations of the cosines. It is one of the objects of special function theory to extend both the formal properties of the cosines and the harmonic analysis for Fourier-cosine series or integrals to these classes of special functions.

The identity

$$(1.1)\qquad \cos n(\theta_1-\theta_2) = \cos n\,\theta_1 \cos n\,\theta_2 + \sin n\,\theta_1 \sin n\,\theta_2,$$

for obvious reasons called an addition formula, has the following well-known generalization to Legendre polynomials:

$$(1.2)\qquad P_n(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) =$$

$$= \sum_{k=-n}^{n} \frac{(n-k)!}{(n+k)!} P_n^k(\cos\theta_1)\, P_n^k(\cos\theta_2)\, e^{ik\phi}$$

(cf. [4], Vol. II, §11.4(8)). Both the formulas (1.1) and (1.2) are contained in Gegenbauer's addition formula for ultraspherical polynomials

$$(1.3)\qquad C_n^\lambda(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) =$$

$$= \sum_{k=0}^{n} a_{n,k}^\lambda (\sin\theta_1)^k\, C_{n-k}^{\lambda+k}(\cos\theta_1)\, (\sin\theta_2)^k\, C_{n-k}^{\lambda+k}(\cos\theta_2)\, \cdot$$

$$\cdot\, C_k^{\lambda-\frac{1}{2}}(\cos\phi), \qquad a_{n,k}^\lambda \text{ constants}$$

(cf. [4], Vol. I, §3.15.1 (19)). Although there exist analytic proofs of formula (1.3), it is in the context of spherical harmonics that this formula is most easily proved and understood (see §2).

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ form a larger and more natural class of orthogonal polynomials than Gegenbauer polynomials (cf. Askey [1]). For $\alpha > -1$, $\beta > -1$ they are orthogonal on the interval $(-1,+1)$ with respect to the weight function $(1-x)^\alpha\,(1+x)^\beta$. We mention the special cases $P_n^{(\alpha,\alpha)}(x) = \text{const.}\; C_n^{\alpha+\frac{1}{2}}(x)$, $P_n^{(0,0)}(x) = P_n(x)$ and

$P_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta) = \text{const.}\cos n\theta$. Departing from formula (1.3) it is difficult to guess what the generalization for Jacobi polynomials might be. One might expect a formula like

$$P_n^{(\alpha,\beta)}(\Lambda(x,y,t)) = \sum_{k=0}^{n} a_{n,k}\, F_n^k(x)\, F_n^k(y)\, p_k(t),$$

where the orthogonal polynomials $p_k(t)$ and the argument $\Lambda(x,y,t)$ have to be specified, and where the functions $F_n^k(x)$ are certain "associated Jacobi functions" such that $F_n^0(x) = P_n^{(\alpha,\beta)}(x)$. However, by group theoretic methods ([10],[11]) we obtained

$$P_n^{(\alpha,\beta)}(\cos 2\Theta(\theta_1,\theta_2,r,\phi)) = \tag{1.4}$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{k} a_{n,k,l}\, F_n^{k,l}(\cos 2\theta_1)\, F_n^{k,l}(\cos 2\theta_2)\, p_{k,l}(r,\phi),$$

where $\cos\Theta(\theta_1,\theta_2,r,\phi) = (\cos\theta_1\cos\theta_2)^2 +$

$$+(\sin\theta_1\sin\theta_2\, r)^2 + \tfrac{1}{2}\sin 2\theta_1\sin 2\theta_2\, r\cos\phi,$$

$$p_{k,l}(r,\phi) = r^{k-l}\, P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\, C_{k-l}^{\beta}(\cos\phi),$$

$$F_n^{k,l}(\cos 2\theta) = (\sin\theta)^{k+l}\,(\cos\theta)^{k-l}\, P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos 2\theta),$$

and where the constants $a_{n,k,l}$ are given in [10]. If we consider $r$ and $\phi$ as polar coordinates then for $\alpha > \beta > -\frac{1}{2}$ the functions $p_{k,l}(r,\phi)$ are orthogonal polynomials in the upper half unit disk with respect to the weight function $(1-x^2-y^2)^{\alpha-\beta-1}\, y^{2\beta}$. Formula (1.4) contains (1.3) as a special (degenerate) case for $\alpha = \beta$, $r = 1$. Our result is essentially more complicated than (1.3) because of the double summation. This phenomenon was completely unexpected from an analytic point of view. Without the group theoretic approach the problem of finding the addition formula for Jacobi polynomials might have remained an open question for many years to come.

In [11] we solved the problem by studying the analogues of spherical harmonics on the homogeneous space $U(q)/U(q-1)$. Thus we obtained (1.4) for $\alpha = 1,2,3,\ldots$ and $\beta = 0$ and we derived the general case by simple analytic methods. In the present paper we give an interpretation of (1.4) in terms of spherical harmonics, when $\alpha$ and $\beta$ are integer or half integer and $\alpha \geq \beta \geq -\frac{1}{2}$. In our opinion this method of proof is more satisfying than our first approach, because it is valid for more general $\alpha$ and $\beta$ and because it is a very elementary method, only using spheres, rotations and spherical harmonics.

## 2. Spherical harmonics

In this section we state without proofs the properties of spherical harmonics which we will need. For the proofs the reader is referred to Erdélyi ([4], Vol. II, Ch. 11), Müller [12] and Vilenkin ([13], Ch. 9).

Let $R^q$ be a q-dimensional real linear vector space with inner product $(x,y)$ and with orthonormal base $e_1,\ldots,e_q$. $SO(q)$ denotes the group of rotations of $R^q$ and $SO(q-1)$ is the subgroup of rotations which leave $e_1$ fixed. Let $\Omega$ be the unit sphere in $R^q$ with rotation invariant measure $d\omega$ and with total measure $\omega_q$. Then $\Omega$ is the homogeneous space $SO(q)/SO(q-1)$.

Let $H(x) = H(x_1,\ldots,x_q)$ be a homogeneous polynomial of degree n on $R^q$ which satisfies Laplace's equation. Let the function $S(\xi)$ be defined on $\Omega$ by $S(\xi) = H(\xi)$ for $\xi \in \Omega$. Then $S(\xi)$ is called a spherical harmonic of degree n. We denote the class of all such functions $S(\xi)$ by $H^{q,n}$ and we write $N(q,n)$ for the dimension of $H^{q,n}$.

THEOREM 2.1. The function space $L^2(\Omega)$ is the direct sum of the spaces $H^{q,n}$ $(n=0,1,2,\ldots)$. The spaces $H^{q,n}$ are mutually orthogonal and they are invariant and irreducible under $SO(q)$.

THEOREM 2.2. Let $S(\xi)$ be a function on $\Omega$. Then $S \in H^{q,n}$ and $S(T\xi) = S(\xi)$ for all $T \in SO(q-1)$ if and only if

$$S(\xi) = \text{const.}\ C_n^{\frac{1}{2}q-1}((\xi,e_1)).$$

COROLLARY 2.3. Let the functions $S_k(\xi)$ $(k=1,\ldots,N(q,n))$ form an orthonormal base of $H^{q,n}$. Then

$$(2.1) \qquad \sum_{k=1}^{N(q,n)} S_k(\xi)\,\overline{S_k(\eta)} = \text{const.}\ C_n^{\frac{1}{2}q-1}((\xi,\eta)), \quad \xi,\eta \in \Omega.$$

The constant equals $N(q,n)\,(\omega_q C_n^{\frac{1}{2}q-1}(1))^{-1}$.

The function $C_n^{\frac{1}{2}q-1}((\xi,\eta))$ can be considered as the kernel function of the function space $H^{q,n}$. We will specify the orthonormal base by using the theorem below. Let $\Omega'$ be the orthoplement of $e_1$ in $\Omega$.

THEOREM 2.4. $H^{q,n}$ is the direct sum of subspaces $H^{q,n,l}$ $(l=0,1,\ldots,n)$ which are mutually orthogonal and which are invariant and irreducible under SO(q-1). $H^{q,n,l}$ consists of the functions

$$S(\xi) = (\sin\theta)^l\, C_{n-l}^{\frac{1}{2}q-1+l}(\cos\theta)\, S_l'(\xi')$$

where $\xi \in \Omega$, $\xi = \cos\theta\, e_1 + \sin\theta\, \xi'$, $0 \le \theta \le \pi$, $\xi' \in \Omega'$ and $S_l' \in H^{q-1,l}$.

COROLLARY 2.5. Let the functions $S_{k_l}(\xi)$ $(k_l=1,\ldots,N(q-1,l))$ form an orthonormal base of $H^{q,n,l}$. Then

$$(2.2) \qquad \sum_{k_l=1}^{N(q-1,l)} S_{k_l}(\xi)\,\overline{S_{k_l}(\eta)} = \text{const.}(\sin\theta_1)^l\, C_{n-l}^{\frac{1}{2}q-1+l}(\cos\theta_1)\,\cdot$$

$$\cdot\,(\sin\theta_2)^l\, C_{n-l}^{\frac{1}{2}q-1+l}(\cos\theta_2)\, C_l^{\frac{1}{2}q-3/2}((\xi',\eta')),$$

where $\xi,\eta \in \Omega$, $\xi = \cos\theta_1\, e_1 + \sin\theta_1\, \xi'$, $\eta = \cos\theta_2\, e_1 + \sin\theta_2\, \eta'$, $0 \le \theta_1 \le \pi$, $0 \le \theta_2 \le \pi$, $\xi' \in \Omega'$, $\eta' \in \Omega'$.

The addition formula (1.3) now follows from (2.1) and (2.2) for $\lambda = \frac{1}{2}q-1$ and $\cos\phi = (\xi',\eta')$. This method will be generalized in section 3. We conclude this section with a lemma which will be applied in section 5.

LEMMA 2.6. Let $S_1 \in H^{q,m}$, $S_2 \in H^{q,n}$, $\xi$ and $\eta \in \Omega$, and let dT the invariant measure on SO(q). Then

$$\int_{SO(q)} S_1(T\xi)\, S_2(T\eta)\, dT = 0 \quad \text{for } m \neq n \text{ and}$$

$$= \lambda\, C_n^{\frac{1}{2}q-1}((\xi,\eta)) \quad \text{for } m = n.$$

Here $$\lambda = (\omega_q\, C_n^{\frac{1}{2}q-1}(1))^{-1} \int_{\Omega_q} S_1(\xi)\, S_2(\xi)\, d\omega(\xi).$$

*Proof*. Denote the left hand side by $F(\xi,\eta)$. $F$ is in $H^{q,m}$ as a function of $\xi$ and $F$ is in $H^{q,n}$ as a function of $\eta$. Since $F(T\xi,T\eta) = F(\xi,\eta)$ for $T \in SO(q)$, $F$ only depends on the inner product $(\xi,\eta)$. By theorem 2.2 $F(\xi,\eta) = \text{const.}\ C_n^{\frac{1}{2}q-1}((\xi,\eta))$ and $F(\xi,\eta) = \text{const.}\ C_m^{\frac{1}{2}q-1}((\xi,\eta))$. Hence $F = 0$ for $m \neq n$. For $m = n$ the constant $\lambda$ is obtained by putting $\xi = \eta$. This completes the proof.

## 3. Jacobi polynomials as spherical functions on symmetric spaces

Let a Riemannian manifold $X$ have the property that for any two point pairs $\xi_1,\xi_2 \in X$, $\eta_1,\eta_2 \in X$ satisfying $d(\xi_1,\xi_2) = d(\eta_1,\eta_2)$ there exists an isometry $T$ of $X$ such that $T\xi_1 = \eta_1$ and $T\xi_2 = \eta_2$. Then $X$ is called a two-point homogeneous space. According to Wang [14] the compact spaces of this type are the spheres $S^d$ $(d=1,2,\ldots)$, the real projective spaces $P^d(R)$ $(d=2,3,\ldots)$, the complex projective spaces $P^d(C)$ $(d=4,6,\ldots)$, the quaternion projective spaces $P^d(H)$ $(d=8,12,\ldots)$ and the Cayley projective plane $P^d(Cay)$ $(d=16)$, where $d$ is the real dimension of the space. These spaces are the compact symmetric spaces of rank one (see Helgason [7]).

Let $X$ be a compact two-point homogeneous space and let $e \in X$ fixed. Let $G$ be the maximal connected group of isometries of $X$ and let $K = \{T \in G \mid Te = e\}$. Then $X$ is the homogeneous space $G/K$. The spherical functions on $X$ are the eigenfunctions of the Laplace-Beltrami operator which only depend on the distance $d(\xi,e)$ $(\xi \in X)$. These functions turn out to be Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos \lambda\, d(\xi,e))$, where $\alpha = (d-2)/2$, $\beta = \alpha$ for $S^d$ and $\beta = -\frac{1}{2},0,1,3$ for $P^d(R)$, $P^d(C)$, $P^d(H)$ and $P^d(Cay)$, respectively (cf. Gangolli [6]). By renormalization of the distance we can put $\lambda = 1$.

The function space $L^2(X)$ is the direct sum of subspaces $H^n$, where $H^n$ is invariant and irreducible under $G$ and contains the spherical function of degree $n$. For an orthonormal base of $H^n$ consisting of functions $S_k(\xi)$ $(k=1,\ldots,N)$ we have

$$(3.1) \qquad \sum_{k=1}^{N} S_k(\xi)\, \overline{S_k(\eta)} = \text{const. } P_n^{(\alpha,\beta)}(\cos d(\xi,\eta)).$$

It can be expected that the analogues of theorem 2.4 and corollary 2.5 will give a generalization of the addition formula (1.3) for those Jacobi polynomials which are spherical functions. The results for $P^d(R)$ follow from (1.3) after a simple quadratic transformation. We obtained the results for $P^d(C)$ as follows.

The complex projective space $P^{2q-2}(C)$ is the homogeneous space $SU(q)/U(q-1)$. Observe that the homogeneous space $U(q)/U(q-1)$ is the unit sphere $\Omega$ in the complex vector space $C^q$ with Hermitian inner product $(z,w) = z_1\bar{w}_1 + \ldots + z_q\bar{w}_q$. Hence the functions on $SU(q)/U(q-1)$ are those functions on $\Omega$ which are invariant under scalar multiplication with $e^{i\phi}$ $(0 \leq \phi < 2\pi)$. Let $H_0^{2q,2n}$ be the class of spherical harmonics $S$ on $\Omega$ which satisfy $S \in H^{2q,2n}$ and $S(e^{i\phi}\xi) = S(\xi)$, $0 \leq \phi < 2\pi$. For an orthonormal base $\{S_k\}$ of $H_0^{2q,2n}$ we have ([11])

$$(3.2) \qquad \sum_{k=1}^{N} S_k(\xi)\, \overline{S_k(\eta)} = \text{const. } P_n^{(q-2,0)}(2|(\xi,\eta)|^2-1), \qquad \xi,\eta \in \Omega.$$

Ikeda and Kayama ([8], [9]) studied the analogues of spherical harmonics for the homogeneous space $U(q)/U(q-1)$. Using their results we were able to prove the analogues of theorem 2.4 and corollary 2.5 for this case. Thus we obtained the addition formula (1.4) for $\alpha = 1,2,3,\ldots$ and $\beta = 0$. The general case of (1.4) then follows by repreated differentiation of both sides of (1.4) with respect to $\phi$ and by analytic continuation with respect to $\alpha$ and $\beta$. Although we did not verify the results for $P^d(H)$ and $P^{16}(Cay)$, these cases probably give formula (1.4) for $\alpha = 3,5,7,\ldots$, $\beta = 1$ and $\alpha = 7$, $\beta = 3$, respectively.

In the context of symmetric spaces of rank one we can give a slightly different interpretation of the addition formula (1.4). It follows from (1.4) by integration that for $\alpha > \beta > -\frac{1}{2}$

$$(3.3) \qquad P_n^{(\alpha,\beta)}(\cos 2\theta_1)\, P_n^{(\alpha,\beta)}(\cos 2\theta_2) =$$

$$= \text{const.} \int_0^1 \int_0^\pi P_n^{(\alpha,\beta)}(\cos 2\Theta)\, (1-r^2)^{\alpha-\beta-1}\, r^{2\beta+1}\, (\sin \phi)^{2\beta}\, dr\, d\phi.$$

On the other hand, spherical functions on a homogeneous space $G/K$ ($K$ compact) satisfy

$$(3.4) \qquad \int_K f(xky)\, dk = f(x)\, f(y), \qquad x,y \in G,$$

cf. Helgason [7], p. 399. Formula (3.3) is the analytic form of (3.4), when $P_n^{(\alpha,\beta)}$ can be interpreted as a spherical function. The addition formula (1.4) can be considered as an orthogonal expansion and its first term is given by (3.3). On the other hand, formula (3.4) gives the first term of an orthogonal expansion of $f(xky)$ as a function of $k \in K$. For compact symmetric spaces of rank one this last interpretation of an addition formula is equivalent to the method described earlier.

## 4. Jacobi polynomials as spherical harmonics

In §3 we considered Jacobi polynomials as spherical functions, i.e. functions on a group $G$ which are bi-invariant with respect to a subgroup $K$. In this section we will obtain a more general class of Jacobi polynomials as functions on a group $G$ which are right invariant under a subgroup $K$ and left invariant under a subgroup $H$.

Let $R^{q+p}$ be a $(q+p)$-dimensional linear vector space with inner product and with orthonormal base $e_1, e_2, \ldots, e_{q+p}$. $\Omega$ denotes the unit sphere in $R^{q+p}$ and $SO(q+p)$ the group of rotations. $\mathcal{H}^{q+p,m}$ is the class of spherical harmonics of degree $m$ on $\Omega$. Suppose that $q > 1$ and $p > 1$. The subspace spanned by $e_1, \ldots, e_q$ is called $R^q$ and the subspace spanned by $e_{q+1}, \ldots, e_{q+p}$ is called $R^p$. The subgroup $SO(q)$ consists of the rotations which leave $e_{q+1}, \ldots, e_{q+p}$ fixed and $SO(p)$ is similarly the subgroup of rotations which leave $e_1, \ldots, e_q$ fixed. We write $\Omega' = \Omega \cap R^q$ and $\Omega'' = \Omega \cap R^p$. If $\xi \in \Omega$ then

$$(4.1) \qquad \xi = \cos\theta\, \xi' + \sin\theta\, \xi'', \quad 0 \le \theta \le \pi/2, \ \xi' \in \Omega', \ \xi'' \in \Omega''.$$

If $d\omega$, $d\omega'$ and $d\omega''$ are the invariant measures on $\Omega$, $\Omega'$ and $\Omega''$, respectively, then it follows from (4.1) that

$$(4.2) \qquad d\omega(\xi) = (\cos\,\theta)^{q-1}\,(\sin\,\theta)^{p-1}\,d\theta\,d\omega'(\xi')\,d\omega''(\xi'').$$

The following theorem is due to Zernike and Brinkman [15] (see also Braaksma and Meulenbeld [2]).

THEOREM 4.1. Let $S(\xi)$ be a function on $\Omega$. Then $S \in H^{q+p,2n}$ and $S(T\xi) = S(\xi)$ for all $T \in SO(q) \times SO(p)$ if and only if $S(\xi) = \text{const.}\ P_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(\cos 2\theta)$.

Next we prove a theorem, which is analogous to theorem 2.4.

THEOREM 4.2. Let $k,l$ integers $\geq 0$ such that $m-k-l$ is even and $\geq 0$. Let $H_{k,l}^{q+p,m}$ be the subclass of $H^{q+p,m}$ which is spanned by the functions

$$(4.3) \qquad S(\xi) = (\cos\,\theta)^{l}\,S_l'(\xi')\,(\sin\,\theta)^{k}\,S_k''(\xi'')\,P_{\frac{1}{2}(m-k-l)}^{(\frac{1}{2}p-1+k,\frac{1}{2}q-1+l)}(\cos 2\theta),$$

where $S_l' \in H^{q,l}$ and $S_k'' \in H^{p,k}$. Then $H^{q+p,m}$ is the direct sum of the spaces $H_{k,l}^{q+p,m}$. The spaces $H_{k,l}^{q+p,m}$ are mutually orthogonal and they are invariant and irreducible under $SO(q) \times SO(p)$.

*Proof.* The invariance, irreducibility and orthogonality follow from theorem 2.1. For the dimension $N(q,n)$ of $H^{q,n}$ holds $(1+x)(1-x)^{1-q} = \sum_{n=0}^{\infty} N(q,n)\,x^n$, $|x| < 1$ (cf. Müller [12]). It follows easily from this formula that $N(q+p,m) = \sum N(q,l)\,N(p,k)$, where the summation is taken over all $k,l \geq 0$ such that $m-k-l \geq 0$ and even. Hence $\dim H^{q+p,m} = \sum \dim H_{k,l}^{q+p,m}$. It is only left to prove that the functions $S(\xi)$ in (4.3) belong to $H^{q+p,m}$.

First observe that the homogeneous polynomial of degree $m+k+l$ given by

$$H(x) = (x_1^2+\ldots+x_{q+p}^2)^{\frac{1}{2}(m+k+l)}\,P_{\frac{1}{2}(m+k+l)}^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}\left(\frac{x_1^2+\ldots+x_q^2-x_{q+1}^2-\ldots-x_{q+p}^2}{x_1^2+\ldots+x_q^2+x_{q+1}^2+\ldots+x_{q+p}^2}\right)$$

is harmonic (by theorem 4.1). Let $a_1,a_2,\ldots,a_{q+p}$ be complex numbers such that $a_1^2 + \ldots + a_q^2 = a_{q+1}^2 + \ldots + a_{q+p}^2 = 0$. Put $u = x_1^2 + \ldots + x_q^2$,

$v = x_{q+1}^2 + \ldots + x_{q+p}^2$. Then it follows easily that

$$(a_1 \frac{\partial}{\partial x_1} + \ldots + a_q \frac{\partial}{\partial x_q})^l \, (a_{q+1} \frac{\partial}{\partial x_{q+1}} + \ldots + a_{q+p} \frac{\partial}{\partial x_{q+p}})^k \, H(x) =$$

$$= 2^{k+l} \, (a_1x_1+\ldots+a_qx_q)^l \, (a_{q+1}x_{q+1}+\ldots+a_{q+p}x_{q+p})^k \, \cdot$$

$$\cdot \, (\frac{\partial}{\partial u})^l \, (\frac{\partial}{\partial v})^k \, [(u+v)^{\frac{1}{2}(m+k+l)} \, P_{\frac{1}{2}(m+k+l)}^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(\frac{u-v}{u+v})].$$

Clearly the resulting function is a harmonic homogeneous polynomial of degree m.

By [4], Vol. II, § 10.8 (16) it follows that

$$(u+v)^n \, P_n^{(\alpha,\beta)} \, (\frac{u-v}{u+v}) \, / \, P_n^{(\alpha,\beta)}(1) =$$

$$= u^n \, {}_2F_1(-n,-n-\beta;\, \alpha+1;\, -\frac{v}{u}) =$$

$$= \sum_{i=0}^n \frac{(-n)_i(-n-\beta)_i}{(\alpha+1)_i \; i!} \, (-1)^i \, u^{n-i} \, v^i.$$

Hence by termwise differentiation

$$(4.4) \qquad (\frac{\partial}{\partial u})^l \, (\frac{\partial}{\partial v})^k \, [(u+v)^n \, P_n^{(\alpha,\beta)}(\frac{u-v}{u+v})] =$$

$$\text{const.} \, (u+v)^{n-k-l} \, P_{n-k-l}^{(\alpha+k,\beta+l)}(\frac{u-v}{u+v}), \quad \text{const.} \neq 0.$$

We conclude that the polynomial

$$(4.5) \; (a_1x_1+\ldots+a_qx_q)^l \, (a_{q+1}x_{q+1}+\ldots+a_{q+p}x_{q+p})^k \, \cdot$$

$$\cdot \, (x_1^2+\ldots+x_{q+p}^2)^{\frac{1}{2}(m-k-l)} \, P_{\frac{1}{2}(m-k-l)}^{(\frac{1}{2}p-1+k,\frac{1}{2}q-1+l)}(\frac{x_1^2+\ldots+x_q^2-x_{q+1}^2-\ldots-x_{q+p}^2}{x_1^2+\ldots+x_q^2+x_{q+1}^2+\ldots+x_{q+p}^2})$$

is harmonic.

The polynomial $(a_1x_1+\ldots+a_qx_q)^l$ is harmonic on $R^q$. Since $H^{q,l}$ is irreducible under $SO(q)$, every harmonic homogeneous polynomial of degree $l$ on $R^q$ is a linear combination of polynomials $(a_1x_1+\ldots+a_qx_q)^l$, where $a_1^2 + \ldots + a_q^2 = 0$. Similar results hold for $(a_{q+1}x_{q+1}+\ldots+a_{q+p}x_{q+p})^k$. By restricting $x$ to $\Omega$ in (4.5) it follows that the functions $S$ defined by (4.3) belong to $H^{q+p,m}$. This completes the proof of theorem 4.2.

COROLLARY 4.3. Let the functions $S_i(\xi)$ $(i=1,\ldots,N(q,l)N(p,k))$ form an orthonormal base of $H^{q+p,m}_{k,l}$. Then

$$\sum_{i=1}^{N(q,l)N(p,k)} S_i(\xi)\,\overline{S_i(\eta)} = \tag{4.6}$$

$$= \text{const.}\; f_m^{k,l}(\cos 2\theta_1)\, f_m^{k,l}(\cos 2\theta_2)\, C_l^{\frac{1}{2}q-1}((\xi',\eta'))\,\cdot$$

$$\cdot\; C_k^{\frac{1}{2}p-1}((\xi'',\eta'')),$$

where $$f_m^{k,l}(\cos 2\theta) = (\cos\theta)^l\,(\sin\theta)^k\,P_{\frac{1}{2}(m-k-l)}^{(\frac{1}{2}p-1+k,\frac{1}{2}q-1+l)}(\cos 2\theta)$$

and $$\xi = \cos\theta_1\,\xi' + \sin\theta_1\,\xi'', \qquad \eta = \cos\theta_2\,\eta' + \sin\theta_2\,\eta''$$
as in formula (4.1).

Combining (2.1) and (4.6) we obtain the formula

$$C_m^{\frac{1}{2}(q+p)-1}(\cos\theta_1\cos\theta_2\cos\phi + \sin\theta_1\sin\theta_2\cos\psi) = \tag{4.7}$$

$$= \sum c_{m,k,l}\, f_{m,k,l}(\cos 2\theta_1)\, f_{m,k,l}(\cos 2\theta_2)\,\cdot$$

$$\cdot\; C_l^{\frac{1}{2}q-1}(\cos\phi)\, C_k^{\frac{1}{2}p-1}(\cos\psi),$$

where the summation is taken over all $k,l \geq 0$ such that $m-k-l$ is even and $\geq 0$, and where $c_{m,k,l}$ are constants. This formula is a generalization of Gegenbauer's addition formula (1.3), but it is not the addition formula for Jacobi polynomials we are looking for.

## 5. The addition formula for Jacobi polynomials

In this section we use the same notation as in section 4. Our approach is to find an interpretation of the product formula (3.3) in terms of spherical harmonics and next to obtain the addition formula as the orthogonal expansion corresponding to this product formula. Suppose that $p > q \geq 2$. The easier cases $p = q$ and $q = 1$ are left to the reader. We will write

$$S(\xi) = P_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(\cos 2\theta), \tag{5.1}$$

where $\theta$ is given by (4.1).

Let $SO(q+p-1)$ consist of the rotations of $R^{q+p}$ which leave $e_1$ fixed. Formula (3.4) applied to the homogeneous space $SO(q+p)/SO(q+p-1)$ gives

$$\int_{SO(q+p-1)} C_m^{\frac{1}{2}(p+q)-1}((T\xi,\eta))\, dT = \tag{5.2}$$
$$= \text{const. } C_m^{\frac{1}{2}(p+q)-1}(\xi)\, C_m^{\frac{1}{2}(p+q)-1}(\eta), \qquad \xi,\eta \in \Omega.$$

If the integration is taken on the subgroup $SO(q) \times SO(p)$ instead of $SO(q+p-1)$ then we obtain for $m = 2n$

$$\int_{SO(q)\times SO(p)} C_{2n}^{\frac{1}{2}(p+q)-1}((T\xi,\eta))\, dT = \text{const. } S(\xi)\, S(\eta). \tag{5.3}$$

This product formula was reduced to an analytic form in [3]. The corresponding orthogonal expansion is the addition formula (4.7).

Next we want to replace the integrand of (5.3) by the function $S$. Let the rotation $A_\tau$ be given by

$$\begin{cases} A_\tau\, e_k = \cos\tau\, e_k - \sin\tau\, e_{q+k} & (k=1,\ldots,q), \\ A_\tau\, e_{q+k} = \sin\tau\, e_k + \cos\tau\, e_{q+k} & (k=1,\ldots,q), \\ A_\tau\, e_k = e_k & (k=2q+1,\ldots,q+p). \end{cases} \tag{5.4}$$

We will prove the product formula

$$(5.5)\qquad \int_{SO(q)\times SO(p)} S(A_\tau T\xi)\, dT = \text{const. } S(A_\tau e_1)\, S(\xi), \quad 0 \le \tau \le \pi/2.$$

Putting $\xi = A_\theta\, e_1$ in (5.5) we obtain the more symmetric formula

$$(5.6)\qquad \int_{SO(q)\times SO(p)} S(A_\tau T A_\theta e_1)\, dT =$$

$$= \text{const. } S(A_\tau e_1)\, S(A_\theta e_1) =$$

$$= \text{const. } P_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(\cos 2\tau)\, P_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(\cos 2\theta),$$

$$0 \le \tau \le \pi/2,\ 0 \le \theta \le \pi/2.$$

Formula (5.5) is essentially due to Flensted-Jensen [5], who first derived the analogue of (5.5) in the dual (i.e. non-compact) case. The product formula (3.3) can be derived from (5.5).

The choice of the rotations $A_\tau$ is motivated by a generalized Cartan decomposition of $SO(q+p)$. If $T \in SO(q+p)$ then $T = T_1\, A_\tau\, T_2$, where $T_1 \in SO(q) \times SO(p)$, $T_2 \in SO(q+p-1)$, $0 \le \tau \le \pi/2$, and where $A_\tau$ is uniquely determined by $T$. Let the group $M$ consist of all $T \in SO(q) \times SO(p)$ for which $TA_\tau = A_\tau T$ $(0 \le \tau \le \pi/2)$. Then $T \in M$ if and only if

$$T = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & T_2 \end{bmatrix},$$

where $T_1$ and $T_2$ are orthogonal matrices with determinant one of orders $q \times q$ and $(p-q) \times (p-q)$, respectively.

For the proof of (5.5), note that the left hand side, considered as a function of $\xi$, satisfies the conditions of theorem 4.1. Hence the left hand side equals $f(\tau)\, S(\xi)$, where $f(\tau) = \text{const.} \int_{SO(q)\times SO(p)} S(A_\tau T e_1) dT$.

Every $T \in SO(q) \times SO(p)$ can be written as $T = T_1\, T_2$, where $T_1 \in M$ and

$T_2 \in SO(q+p-1)$. Thus $S(A_\tau Te_1) = S(A_\tau T_1 T_2 e_1) = S(T_1 A_\tau e_1) = S(A_\tau e_1)$. This proves formula (5.5).

We will derive the addition formula for Jacobi polynomials by expanding $S(A_\tau T\xi)$ as a function of $T \in SO(q) \times SO(p)$. By theorem 4.2 it is an equivalent problem to find the expansion

$$(5.5) \qquad S(A_\tau \xi) = \sum_{k=0}^{n} \sum_{l=-k}^{+k} S_{k,l}(\xi,\tau),$$

where $S_{k,l}(\xi,\tau)$ belongs to $H^{q+p,2n}_{k+l,k-l}$ as a function of $\xi$. Since $S(A_\tau T\xi) = S(A_\tau \xi)$ for $T \in M$, we have the invariance $S_{k,l}(T\xi,\tau) = S_{k,l}(\xi,\tau)$ for $T \in M$.

Let $R^q_0$ be the subspace of $R^{q+p}$ which is spanned by $e_{q+1},\dots,e_{2q}$ and let $R^{p-q}_0$ be spanned by $e_{2q+1},\dots,e_{q+p}$. Let $\Omega'_0 = \Omega \cap R^q_0$ and $\Omega''_0 = \Omega \cap R^{p-q}_0$. Let the linear mapping $I\colon R^q_0 \to R^q$ be defined by $I\, e_{q+k} = I\, e_k$ $(k=1,\dots,q)$. Similarly to (4.1) we can write for $\xi'' \in \Omega''$:

$$(5.6) \qquad \xi'' = \cos\chi\, \xi'_0 + \sin\chi\, \xi''_0, \quad 0 \le \chi \le \pi/2,\ \xi'_0 \in \Omega'_0,\ \xi''_0 \in \Omega''_0.$$

LEMMA 5.1. The M-invariant functions in $H^{q+p,2n}_{k+l,k-l}$ are zero for $l < 0$ and they are equal to

$$(5.7) \qquad S(\xi) = \text{const.}(\sin\theta)^{k+l} (\cos\theta)^{k-l} P_{n-k}^{(\frac{1}{2}p-1+k+l,\frac{1}{2}q-1+k-l)}(\cos 2\theta)\ \cdot$$

$$\cdot\ (\cos\chi)^{k-l} P_l^{(\frac{1}{2}(p-q)-1,\frac{1}{2}q-1+k-l)}(\cos 2\chi)\, C_{k-l}^{\frac{1}{2}q-1}((\xi',I\xi'_0))$$

for $l \ge 0$. Here the notation of (4.1) and (5.6) is used.

*Proof*. The M-invariant functions in $H^{q+p,2n}_{k+l,k-l}$ are obtained by the projection $S_0(\xi) = \int_M S(T\xi)\, dT$, $S \in H^{q+p,2n}_{k+l,k-l}$. By applying theorem 4.2 twice we can derive that $H^{q+p,2n}_{k+l,k-l}$ is spanned by the functions

$$S(\xi) = (\sin\theta)^{k+l}\ (\cos\theta)^{k-l}\ P_{n-k}^{(\frac{1}{2}p-1+k+l,\frac{1}{2}q-1+k-l)}(\cos 2\theta)\ \cdot$$

$$\cdot\ (\sin\chi)^{i}\ (\cos\chi)^{j}\ P_{\frac{1}{2}(k+l-i-j)}^{\frac{1}{2}(p-q)-1+i,\frac{1}{2}q-1+j)}(\cos 2\chi)\ \cdot$$

$$\cdot\ S'_{k-l}(\xi')\ S'_{0j}(\xi'_0)\ S''_{0i}(\xi''_0),$$

where $0 \le k \le n$, $-k \le l \le k$, $i \ge 0$, $j \ge 0$, $k+l-i-j \ge 0$ and even, $S'_{k-l} \in H^{q,k-l}$, $S'_{0j} \in H^{q,j}$, $S''_{0i} \in H^{p-q,i}$.
By writing $S(\xi) = f(\theta,\chi,\xi',\xi'_0,\xi''_0)$ the projection $S \to S_0$ becomes

$$S_0(\xi) = \int_{T_1\in SO(q)} \int_{T_2\in SO(p-q)} f(\theta,\chi,T_1\xi',I^{-1}T_1I\xi'_0,T_2\xi''_0)\ dT_1\ dT_2.$$

By applying lemma 2.6 twice it follows that $S_0 = 0$ if $i \neq 0$ or $j \neq k-l$. For $i = 0$ and $j = k-l$ we have $k+l-i-j = 2l$, hence $l \ge 0$. Formula (5.7) now follows easily from lemma 2.6. This completes the proof.

For fixed $p,q,n$ and for $k,l$ such that $0 \le l \le k \le n$ we define

$$(5.8)\quad S_{k,l}(\xi) = c_{k,l}(\sin\theta)^{k+l}\ (\cos\theta)^{k-l}\ P_{n-k}^{(\frac{1}{2}p-1+k+l,\frac{1}{2}q-1+k-l)}(\cos 2\theta)\ \cdot$$

$$\cdot\ (\cos\chi)^{k-l}\ P_{l}^{(\frac{1}{2}(p-q)-1,\frac{1}{2}q-1+k-l)}(\cos 2\chi)\ C_{k-l}^{\frac{1}{2}q-1}((\xi',I\xi'_0)),$$

where the constants $c_{k,l}$ are positive and such that $\int_{\Omega_{q+p}} (S_{k,l}(\xi))^2\ d\omega(\xi) = 1$.
Observe that $S_{0,0}(\xi) = c_{0,0}\ S(\xi)$ and that $S_{k,l}(e_1) = 0$ for $(k,l) \neq (0,0)$. The functions $S_{k,l}(\xi)$ form an orthonormal base of a function space which is invariant under transformations $A_\tau$. Hence, the kernel function

$$F(\xi,\eta) = \sum_{k=0}^{n} \sum_{l=0}^{k} S_{k,l}(\xi)\ S_{k,l}(\eta),\quad \xi,\eta \in \Omega,$$

satisfies $F(A_\tau\xi,A_\tau\xi) = F(\xi,\eta)$. Putting $\eta = A_\tau^{-1}e_1$ we obtain the required expansion of type (5.5).

THEOREM 5.2.

$$(5.9)\qquad S(A_\tau \xi) = c_{00}^{-1} \sum_{k=0}^{n} \sum_{l=0}^{k} S_{k,l}(A_\tau^{-1} e_1)\, S_{k,l}(\xi),$$

where the functions $S_{k,l}$ are given by (5.8).

Apart from a different notation of the variables, the right hand side of (5.9) is the same as the right hand side of (1.4). The left hand side of (5.9) can be handled as follows. Let $S(A_\tau \xi) = P_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(\cos 2\theta)$. Then

$$\begin{aligned}
\cos^2 \theta &= \sum_{k=1}^{q} ((A_\tau \xi, e_k))^2 = \\
&= \sum_{k=1}^{q} ((\xi, \cos\tau\, e_k + \sin\tau\, e_{q+k}))^2 = \\
&= \sum_{k=1}^{q} [\cos^2\tau\, ((\xi, e_k))^2 + \sin^2\tau\, ((\xi, e_{q+k}))^2 + \\
&+ \sin 2\tau\, (\xi, e_k)\, ((\xi, e_{q+k}))] = \\
&= \sum_{k=1}^{q} [\cos^2\tau \cos^2\theta\, ((\xi', e_k))^2 + \\
&+ \sin^2\tau \sin^2\theta \cos^2\chi\, ((\xi_0', e_{q+k}))^2 + \\
&+ \tfrac{1}{2} \sin 2\tau \sin 2\theta \cos\chi\, (\xi', e_k)\, (\xi_0', e_{q+k})] = \\
&= \cos^2\tau \cos^2\theta + \sin^2\tau \sin^2\theta \cos^2\chi + \\
&+ \tfrac{1}{2} \sin 2\tau \sin 2\theta \cos\chi\, (\xi', I\xi_0').
\end{aligned}$$

This proofs formula (1.4) for $\alpha = \frac{1}{2}p-1$, $\beta = \frac{1}{2}q-1$. We omit the routine calculation of the constant coefficients $c_{k,l}$ in (5.8).

References

[1] R. Askey, Jacobi polynomials, I, New proofs of Koornwinder's Laplace type integral representation and Bateman's bilinear sum, Mathematics Research Center Madison Technical Summary Report 1216 (1972).

[2] B.L.J. Braaksma and B. Meulenbeld, Jacobi polynomials as spherical harmonics, Indag. Math. 30 (1968), 384-389.

[3] A. Dijksma and T.H. Koornwinder, Spherical functions and the product of two Jacobi polynomials, Indag. Math. 33 (1971), 191-196.

[4] A. Erdélyi, et al., Higher transcendental functions, Vol. I, II, McGraw-Hill, New York, 1953.

[5] M. Flensted-Jensen, Spherical functions on rank one symmetric spaces and generalizations, to appear in "Harmonic analysis on homogeneous spaces", Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, Rhode Island.

[6] R. Gangolli, Positive definite kernels on homogeneous spaces and certain stochastic processes related to Levy's Brownian motion of several parameters, Ann. Inst. Henri Poincaré, Sect. B., 3 (1967), 121-226.

[7] S. Helgason, Differential geometry and symmetric spaces, Academic Press, New York, 1962.

[8] M. Ikeda, On spherical functions for the unitary group (I,II,III), Mem. Fac. Engrg. Hiroshima Univ. 3 (1967), 17-75.

[9] ________ and T. Kayama, On spherical functions for the unitary group (IV), ibidem, 77-100.

[10] T.H. Koornwinder, The addition formula for Jacobi polynomials, I, Summary of results, Indag. Math. 34 (1972), 188-191.

[11] ________, The addition formula for Jacobi polynomials, II, III, Math. Centrum Amsterdam Afd. Toegepaste Wisk. Rep. (1972).

[12] C. Müller, Spherical harmonics, Lecture Notes in Math. 17, Springer-Verlag, Berlin, 1966.

[13] N.J. Vilenkin, Special functions and the theory of group representations, Am. Math. Soc. Transl. of Math. Monographs, vol. 22, 1968.

[14] H.C. Wang, Two-point homogeneous spaces, Ann. of Math. 55 (1952), 177-191.

[15] F. Zernike and H.C. Brinkman, Hypersphärische Funktionen und die in sphärischen Bereichen orthogonalen Polynome, Proc. Royal Acad. Amsterdam 38 (1935), 161-170.